Boundary Harnack Principle for Diffusion with Jumps in Lipschitz domain

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The 14th Workshop on Markov Processes and Related Topics

(Joint work with Professor Z.-Q. Chen)

- We say that h is a harmonic function with respect to a process X on an open set D if for each open bounded $V \subset \overline{V} \subset D$,

$$h(x) = \mathbb{E}_x h(X_{\tau_V}), \quad x \in V,$$

where $\tau_V := \inf\{t > 0 : X_t \notin V\}$.

- The statement of the boundary Harnack principle(BHP):
Let D be a domain with a certain geometric property. There exist positive constants R_0 and C such that for any $Q \in \partial D, r \in (0, R_0]$ any positive harmonic functions u and v w.r.t. X in $D \cap B(Q, r)$ that vanish continuously on $D^c \cap B(Q, r)$, we have

$$\frac{u(x)}{u(y)} \leq C \frac{v(x)}{v(y)}, \quad x, y \in D \cap B(Q, r/2).$$

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Background: BHP of B.M. and Diffusion Processes

- The boundary Harnack principle(BHP) for Brownian motion in Lipschitz domain was obtained independently by Ancona (1978), Dahlberg (1977) and Wu (1978).
- The result was generalized to elliptic operator by Caffarelli-Fabes- Mortola-Salsa (1981) and Fabes-Carofalo-Marin-Malave and Salsa (1988).
- In 1989, Bass-Burdzy developed a quite different probabilistic method (also called the "box" method) to prove the BHP of Brownian motion in Lipschitz domain and extended it to more general non-smooth domain.
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Pure Discontinuous Processes

- A stochastic process $Z = (Z_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ is called a rotationally symmetric α-stable process with $\alpha \in (0,2)$ on $\mathbb{R}^d$ if it is a Lévy process such that its characteristic function is given by

$$\mathbb{E}_x\left[e^{i\xi\cdot(Z_t - Z_0)}\right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x \in \mathbb{R}^d.$$

- The infinitesimal generator for rotationally symmetric α-stable process

$$\Delta^{\alpha/2} f(x) = \mathcal{A}(d,-\alpha)\int_{\mathbb{R}^d}\left(f(x+z) - f(x) - \nabla f(x)\cdot z 1_{\{|z|\le 1\}}\right)\frac{dz}{|z|^{d+\alpha}}$$

- Truncated symmetric α-stable process is the symmetric α-stable process with large jumps more than 1 removed. Denote by $\overline{\Delta}^{\alpha/2}$ the operator of truncated symmetric α-stable process, then

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Main Goal

The goal

Our aim is to obtain the BHP for a class of diffusion with jumps in Lipschitz domain.

The setting

Suppose $d \geq 3$ and $0 < \alpha < 2$. Define

$$\mathcal{L}f(x) := \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}f(x)\right) + b(x)\cdot\nabla f(x) + \mathcal{S}^{\kappa}f(x), \quad f \in C_b^2(\mathbb{R}^d)$$

where

$$\mathcal{S}^{\kappa}f(x) := \int_{\mathbb{R}^d}\left(f(x+z) - f(x) - \nabla f(x)\cdot z 1_{\{|z|\leq 1\}}\right)\frac{\kappa(x,z)}{|z|^{d+\alpha}}dz.$$

Assumptions:

(1) Assume that for each $x \in \mathbb{R}^d$, $a_{ij}(x)$ is a symmetric matrix and satisfies the uniform ellipticity condition and the Hölder continuity condition, i.e.

$$L^{-1} I_{d\times d} \le a_{ij}(x) \le L I_{d\times d}.$$

$$|a_{ij}(x_1) - a_{ij}(x_2)| \le c|x_1 - x_2|^{\gamma}, \quad x_1, x_2 \in \mathbb{R}^d.$$

(2) We assume that $b(\cdot)$ belongs to Kato class K_d^1. Here we say that a function $b \in K_d^1$ if

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} |y-x|^{1-d} |b|(y) = 0.$$

(3) $\kappa(x,z)$ is a nonnegative real-valued bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $\kappa(x,z) = \kappa(x,-z)$ for every $x, z \in \mathbb{R}^d$ and there exist $\rho > 0$ and $A \ge 1$ such that

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Remarks:

1. By a similar argument as Z.-Q. Chen-E. Hu-L. Xie-X. Zhang(2016), the fundamental solution of $\mathcal{L}$ exists. Associated with it is a conservative Feller process X on the canonical Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$ having Lévy system $(J^\kappa(x,y)dy,dt)$, where $J^\kappa(x,y)=\kappa(x,y-x)/|y-x|^{d+\alpha}$.
2. In general, the operator $\mathcal{L}$ is a nonsymmetric operator. It covers the two examples as belowing:
   (1) $\mathcal{L}=\Delta+b\cdot\nabla+a\Delta^{\alpha/2}$ with $a_{ij}=I$ and $\kappa(x,z)=a$.
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Assumption $(A1)$:

For each set $F \subseteq \mathbb{R}^d$ and each $u \in F^c$, define

$$\Lambda_{u,F} := \{y \in F : J^\kappa(y,u) > 0\}.$$

For each $x \in \mathbb{R}^d$ and $r > 0$, define

$$\|\kappa\|_{x,r} := \text{esssup}\{\kappa(y, u-y) : \; y \in B(x,r), u \in B(x,4r)^c\}.$$

Assumption (A1): There exist positive constants δ and c such that for any $r \in (0,\delta)$, $x \in \mathbb{R}^d$ and $u \in \{u \in B(x,4r)^c : m(\Lambda_{u,B(x,r)}) > 0\}$,

$$\frac{1}{r^d}\int_{B(x,2r)} J^\kappa(y,u)\,dy \geq cJ^{\|\kappa\|_{x,r}}(w,u), \quad w \in B(x,r).$$

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Theorem:(Harnack principle)

Suppose Assumption $(A1)$ holds. There exist two positive constants $\delta = \delta(d, \alpha)$ and $C = C(d, \alpha, \kappa)$ such that for any $x \in \mathbb{R}^d, r \in (0, \delta)$ and any nonnegative bounded harmonic function h with respect to X in $B(x, r)$,

$$h(y_1) \le Ch(y_2), \quad y_1, y_2 \in B(x, r/4).$$

Assumption (A2):

Suppose D is a Lipschitz domain with characteristic (R, Λ). Fix a constant $\eta \in (0, 1/4)$. For each $r \in (0, R)$ and $Q \in \partial D$, define the "interior set" of $D \cap B(Q, r)$

$$\Omega_{\eta,r,Q} := \{y \in D \cap B(Q, r) : \delta_{D \cap B(Q,r)}(y) > \eta r\}.$$

For each $Q \in \partial D$ and $r \in (0, R)$, define

$$\|\kappa\|_{Q,D,r} := \text{esssup}\{\kappa(y, u - y) : \ y \in D \cap B(Q, r), u \in B(Q, 4r)^c\}.$$

Assumption (A2): There exist two positive constants $R_1 \in (0, R)$ and $c > 0$ such that for any $r \in (0, R_1)$, $Q \in \partial D$ and

$$u \in \left\{ u \in B(Q, 4r)^c : m(\Lambda_{u, D \cap B(Q,r)}) > 0 \right\},$$

we have

$$\frac{1}{r^d} \int_{\Omega_{\eta,2r,Q}} J^{\kappa}(y, u)\, dy \geq c J^{\|\kappa\|_{Q,D,r}}(w, u), \quad w \in D \cap B(Q, r).$$

Remark:

1. The condition $(A2)$ is restricted on large jumps of X from small sets near the boundary of D.
2. Roughly speaking, the assumption $(A2)$ tells that if X has large jumps from $D \cap B(Q, r)$ to $B(Q, 4r)^c$, then it is required that there have enough jumps from the "interior set" $\Omega_{\eta,2r,Q}$ of $D \cap B(Q, 2r)$ to $B(Q, 4r)^c$.

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Chen-W. 2018+ (BHP in Lipschitz domain)

Suppose Assumptions $(A1)$ and $(A2)$ hold. For each Lipschitz open set D with characteristics (R, Λ), there exists a positive constant C such that for all $Q \in \partial D, r \in (0, R]$ and all functions $h_k \geq 0, k = 1, 2$ on $\mathbb{R}^d$ that are harmonic with respect to X in $D \cap B(Q, r)$ and vanish continuously on $D^c \cap B(Q, r)$, we have

$$\frac{h_1(x)}{h_1(y)} \leq C \frac{h_2(x)}{h_2(y)}, \quad x, y \in D \cap B(Q, r/2).$$

Theorem 1: Suppose that $A > 0$. For any Lipschitz domain D with characteristics (R, Λ), there exists a positive constant $C = C(d, \alpha, R, \Lambda, A)$ such that for $Q \in \partial D, r \in (0, R], A^{-1} \leq \kappa(x, z) \leq A$ and all functions $h_k \geq 0, k = 1, 2$ on $\mathbb{R}^d$ that are harmonic with respect to X in $D \cap B(Q, r)$ and vanish continuously on $D^c \cap B(Q, r)$, we have

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Remark:
$\mathcal{L} = \Delta + a\Delta^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a$.

Theorem 2: Suppose that $A > 0$. For each Lipschitz open set D with characteristics (R, Λ), suppose there exists a constant $c = c(d, R, \Lambda) > 0$ such that for any $r \in (0, R)$, $Q \in \partial D$ and $u \in \left\{u \in B(Q, 4r)^c : m(H_{u,D_{Q,r}}) > 0\right\}$,

$$\int_{\Omega_{\kappa,2r,Q}} 1_{|u-y|\le 1}(y)\, dy \ge cr^d.$$

Then there exists a positive constant $C = C(d, \alpha, R, \Lambda, A)$ such that for $Q \in \partial D, r \in (0, R], A^{-1}1_{|z|\le 1}(z) \le \kappa(x, z) \le A1_{|z|\le 1}(z)$ for $x \in \mathbb{R}^d$ and all functions $h_k \ge 0, k = 1, 2$ on $\mathbb{R}^d$ that are harmonic with respect to X in $D \cap B(Q, r)$ and vanish continuously on $D^c \cap B(Q, r)$, we have

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Remark:
$\mathcal{L} = \Delta + a\overline{\Delta}^{\alpha/2}$ with $a_{ij} = I$ and $\kappa(x, z) = a1_{\{|z|\le 1\}}(z)$.

Main Method

Bass-Burdzy (1989) developed a probabilistic method (also called the "box" method) to prove the BHP of Brownian motion in Lipschitz domain. Bogdan(1999) adapted this method in the symmetric stable process.

We shall use this method in our case to prove the BHP in Lipschitz domain.

An open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be Lipschitz domain if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a Lipschitz function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$\phi(0) = \nabla\phi(0) = 0, |\phi(z_1) - \phi(z_2)| \leq \Lambda|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R}^{d-1}$$

and an orthonormal coordinate system $CS_Q : y = (y_1, \cdots, y_{d-1}, y_d) =: (\tilde{y}, y_d)$ with its origin at Q such that

$$B(Q,R) \cap D = \{y = (\tilde{y}, y_d) \in B(0,R) \, in \, CS_Q : y_d > \phi(\tilde{y})\}.$$

The pair (R, Λ) is called the characteristics of the Lipschitz domain D.

"Box" Method

- For each $Q \in \partial D$ and $x \in B(Q, R)$, define the vertical distance from x to ∂D by
$$\rho_Q(x) = x^d - \phi_Q(x^1, \cdots, x^{d-1}),$$
where $(x^1, \cdots, x^d)$ is the coordinate of x in CS_Q.
- For each $Q \in \partial D$, define the "box"
$$\Delta(Q, a, R) = \{y \in CS_Q : 0 < \rho_Q(y) < a, |(y^1, \cdots, y^{d-1})| < R\}.$$
$$\nabla(Q, a, R) = \{y \in CS_Q : -a < \rho_Q(y) < 0, |(y^1, \cdots, y^{d-1})| < R\}.$$

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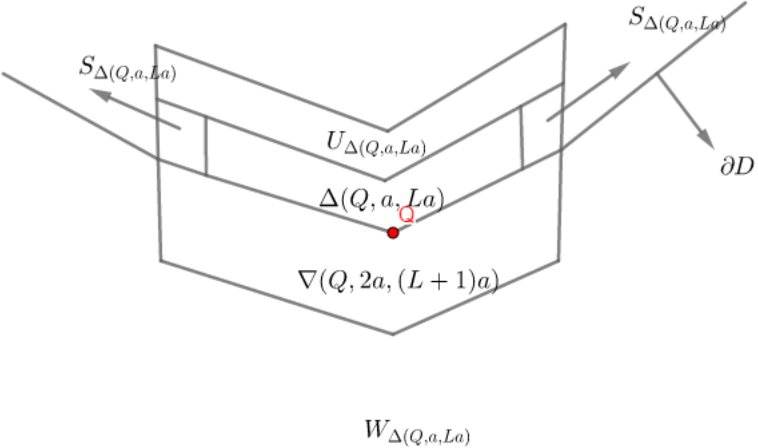
Let $Q \in \partial D$. For the "box" $\Delta(Q, a, La)$, we define

$$S_{\Delta(Q,a,La)} := \Delta(Q, a, (L+1)a) \setminus \Delta(Q, a, La)$$
$$U_{\Delta(Q,a,La)} := \Delta(Q, 2a, (L+1)a) \setminus \Delta(Q, a, (L+1)a)$$
$$W_{\Delta(Q,a,La)} := [\Delta(Q, 2a, (L+1)a) \cup \nabla(Q, 2a, (L+1)a)]^c.$$



$S_{\Delta(Q,a,La)}$
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∂D
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Q
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- Let h be a harmonic function in $\Delta(Q, 4r, 10r)$ and vanishes continuously on $\nabla(Q, 4r, 10r)$. Then for $x \in \Delta(Q, r, r)$,

$$
\begin{aligned}
h(x) =& \mathbb{E}_x h(X_{\tau_{\Delta(Q,2r,8r)}}) \\
=& \mathbb{E}_x[h(X_{\tau_{\Delta(Q,2r,8r)}}); X_{\tau_{\Delta(Q,2r,8r)}} \in S_{\Delta(Q,2r,8r)}] \\
&+ \mathbb{E}_x[h(X_{\tau_{\Delta(Q,2r,8r)}}); X_{\tau_{\Delta(Q,2r,8r)}} \in U_{\Delta(Q,2r,8r)}] \\
&+ \mathbb{E}_x[h(X_{\tau_{\Delta(Q,2r,8r)}}); X_{\tau_{\Delta(Q,2r,8r)}} \in W_{\Delta(Q,2r,8r)}]
\end{aligned}
$$

Main goal:

For $x \in \Delta(Q, r, r)$,

$$h(x) \asymp h(x_0)\mathbb{P}_x(X_{\tau_{\Delta(Q,2r,8r)}} \in U_{\Delta(Q,2r,8r)}),$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$. Thus,

$$\frac{h(x)}{h(y)} \asymp \frac{\mathbb{P}_x(X_{\tau_\Delta} \in U_\Delta)}{\mathbb{P}_y(X_{\tau_\Delta} \in U_\Delta)}, \quad x, y \in \Delta(Q, r, r).$$

That is, the BHP holds.

Proposition:(Key estimates)

There exists $c > 0$ such that for any $Q \in \partial D$ and $x \in \Delta(Q, r, r)$, we have

$$\mathbb{P}_x(X_{\tau_{\Delta(Q,2r,8r)}} \in S_{\Delta(Q,2r,8r)}) \le c\mathbb{P}_x(X_{\tau_{\Delta(Q,2r,8r)}} \in U_{\Delta(Q,2r,8r)}).$$

Lemma 1:

There exist positive constants $c_k, k = 0, 1$ such that for every $Q \in \partial D, a < \delta, L \ge 1$ and $x \in \Delta(Q, a, La/2)$

$$\mathbb{P}_x(X_{\tau_{\Delta(Q,a,La)}} \in S_{\Delta(Q,a,La)}) \le c_1 \exp(-c_0 L) + c_1 a^{2-\alpha}(L+1)^{-(1+\alpha)}.$$

Lemma 2:

There exist positive constants c and β such that for every $Q \in \partial D, a < \delta, L \ge 1$ and $y \in \Delta(Q, a/2, a)$, we have

$$\mathbb{P}_y(X_{\tau_{\Delta(Q,a,La)}} \in U_{\Delta(Q,a,La)}) \ge c\left(\frac{\delta_D(y)}{a}\right)^{\beta} + c(\delta_D(y))^{2-\alpha}\left(\frac{\delta_D(y)}{a}\right)^{\alpha}.$$

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Proposition: (Carleson estimate)

There exists a constant c such that for any $0 < r < R, Q \in \partial D$ and any nonnegative harmonic function h in $D \cap B(Q,r)$ which vanishes in $D^c \cap B(Q,r)$, we have

$$h(x) \le ch(x_0), \quad x \in D \cap B(Q, r/2),$$

where $x_0 \in D \cap B(Q,r)$ with $\delta_D(x_0) = r/2$.

Estimates in small range

Let $\Delta := \Delta(Q, 2r, 8r)$. For $x \in \Delta(Q,r,r)$,

$$\begin{aligned} &\mathbb{E}_x[h(X_{\tau_\Delta}), X_{\tau_\Delta} \in S_\Delta] + \mathbb{E}_x[h(X_{\tau_\Delta}), X_{\tau_\Delta} \in U_\Delta] \\ \le& c_1 h(x_0)(\mathbb{P}_x[X_{\tau_\Delta} \in S_\Delta] + \mathbb{P}_x[X_{\tau_\Delta} \in U_\Delta]) \\ \le& c_2 h(x_0)\mathbb{P}_x[X_{\tau_\Delta} \in U_\Delta]. \end{aligned}$$

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Estimates for large jumps

Suppose D is an open set in $\mathbb{R}^d$ and h is a non-negative function, we have

$$\mathbb{E}_x[h(X_{\tau_D}), X_{\tau_D -} \neq X_{\tau_D}] = \int_{D^c} \int_D G_D^X(x,y) \frac{\kappa(y, u-y)}{|y-u|^{d+\alpha}} dy h(u)\, du.$$

Let $\Delta := \Delta(Q, 2r, 8r)$. Hence, for $x \in \Delta(Q, r, r)$,

$$\mathbb{E}_x[h(X_{\tau_\Delta}), X_{\tau_\Delta} \in W_\Delta] = \int_{W_\Delta} \int_\Delta G_\Delta^X(x,y) \frac{\kappa(y, u-y)}{|y-u|^{d+\alpha}} h(u)\, dy\, du.$$

By the assumption $(A2)$, for $x \in \Delta(Q, r, r)$,

$$\mathbb{E}_x[h(X_{\tau_\Delta}), X_{\tau_\Delta} \in W_\Delta] \leq c h(x_0) \mathbb{P}_x(X_{\tau_\Delta} \in U_\Delta),$$

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Proof of BHP

1. Let h be a harmonic function in $\Delta(Q, 4r, 10r)$ and vanishes continuously on $\nabla(Q, 4r, 10r)$. Let $\Delta := \Delta(Q, 2r, 8r)$. Then for $x \in \Delta(Q, r, r)$,

$$\begin{aligned} h(x) =& \mathbb{E}_x[h(X_{\tau_\Delta}); X_{\tau_\Delta} \in S_\Delta] + \mathbb{E}_x[h(X_{\tau_\Delta}); X_{\tau_\Delta} \in U_\Delta] \\ &+ \mathbb{E}_x[h(X_{\tau_\Delta}); X_{\tau_\Delta} \in W_\Delta] \\ \le& ch(x_0)\mathbb{P}_x(X_{\tau_\Delta} \in U_\Delta), \end{aligned}$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) = r/2$.

2. On the other hand, by the assumption $(A1)$ and the Harnack principle,

$$\begin{aligned} h(x) &\ge \mathbb{E}_x[h(X_{\tau_\Delta}); X_{\tau_\Delta} \in U_\Delta] \\ &\ge ch(x_0)\mathbb{P}_x(X_{\tau_\Delta} \in U_\Delta). \end{aligned}$$

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Thank you!